

Monadic $n \times m$ -valued Łukasiewicz-Moisil algebras

Aldo V. Figallo and Claudia Sanza

Abstract

Here we initiate an investigation into the class $\mathbf{mLM}_{n \times m}$ of monadic $n \times m$ -valued Łukasiewicz-Moisil algebras (or $mLM_{n \times m}$ -algebras), namely $n \times m$ -valued Łukasiewicz-Moisil algebras endowed with a unary operation called existential quantifier. These algebras constitute a generalization of monadic n -valued Łukasiewicz-Moisil algebras. In this article, the relationship between existential quantifiers and special subalgebras of $mLM_{n \times m}$ -algebras is shown. Besides, the congruences on these algebras are determined and subdirectly irreducible algebras are characterized. From this last result it is proved that $\mathbf{mLM}_{n \times m}$ is a discriminator variety and as a consequence, the principal congruences are characterized. Furthermore, the number of congruences of finite $mLM_{n \times m}$ -algebras is computed. In addition, a topological duality for $mLM_{n \times m}$ -algebras is described and a characterization of $mLM_{n \times m}$ -congruences in terms of special subsets of the associated space is shown. Moreover, the subsets which correspond to principal congruences are determined. Finally, some functional representation theorems for these algebras are given and the relationship between them is pointed out.

Key words. n -valued Łukasiewicz-Moisil algebras, monadic n -valued Łukasiewicz-Moisil algebras, congruences, subdirectly irreducible algebras, discriminator variety, Priestley spaces.

1 Introduction and preliminaries

In 1975, W. Suchoń ([20]) introduced matrix Łukasiewicz algebras, so generalizing the notion of n -valued Łukasiewicz algebras without negation ([13]). The only paper about these algebras is the one mentioned above and a brief reference to them can be found in [2]. On the other hand, in [18] we introduced $n \times m$ -valued Łukasiewicz algebras with negation. Later, following the terminology established in [2], they were called $n \times m$ -valued Łukasiewicz-Moisil algebras (or $LM_{n \times m}$ -algebras for short) and since then, we have named them in this way. These algebras are both, a particular case of matrix Łukasiewicz algebras and a generalization of n -valued Łukasiewicz-Moisil algebras ([2]). $LM_{n \times m}$ -algebras were studied in [15], [18], [19] and [8]. In particular, in [18] we provided an important example

which legitimated the study of this new class of algebras. Besides, in [8] we presented a propositional calculus which has $LM_{n \times m}$ -algebras as algebraic counterpart.

In the present paper, we introduce and investigate monadic $n \times m$ -valued Łukasiewicz-Moisil algebras which constitute a generalization of monadic n -valued Łukasiewicz-Moisil algebras ([2, 10]).

The paper is organized as follows. In Section 1, we briefly summarize the main definitions and results needed throughout this article. In Section 2, we introduce monadic $n \times m$ -valued Łukasiewicz-Moisil algebras (or $mLM_{n \times m}$ -algebras), namely $LM_{n \times m}$ -algebras endowed with a unary operation called exsistential quantifier. Besides, we show their most important properties which are necessary for further development. Furthermore, we determine the relationship between existential quantifiers and special subagebras of $LM_{n \times m}$ -algebras. In Section 3, we determine the $mLM_{n \times m}$ -congruences and we characterize subdirectly irreducible algebras. In Section 4, by applying the results obtained in the previous section, we show that this variety is a discriminator variety and as a consequence, we deduce some properties of the congruences. In Section 5, we show a topological duality for $mLM_{n \times m}$ -algebras and we characterize the congruences by means of special subsets of the associated space. In particular, we determine which of these subsets correspond to principal congruences. Finally, in Section 6, we describe three functional representation theorems for $mLM_{n \times m}$ -algebras, pointing out the relationship between them.

We refer the reader to the bibliography listed here as [3, 1, 2, 11, 12, 4, 13, 10] for specific details of the many basic notions and results of universal algebra including distributive lattices, De Morgan algebras, Boolean algebras, monadic Boolean algebras, n -valued Łukasiewicz-Moisil algebras and monadic n -valued Łukasiewicz-Moisil algebras considered in this paper.

An $n \times m$ -valued Łukasiewicz-Moisil algebra, in which n and m are integers, $n \geq 2$, $m \geq 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, 0, 1 \rangle$ where $(n \times m)$ is the cartesian product $\{1, \dots, n-1\} \times \{1, \dots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0, 1 \rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j) \in (n \times m)}$ is a family of unary operations on L which fulfills these conditions:

$$(C1) \quad \sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y,$$

$$(C2) \quad \sigma_{ij}x \leq \sigma_{(i+1)j}x,$$

$$(C3) \quad \sigma_{ij}x \leq \sigma_{i(j+1)}x,$$

$$(C4) \quad \sigma_{ij}\sigma_{rs}x = \sigma_{rs}x,$$

$$(C5) \quad \sigma_{ij}x = \sigma_{ij}y \text{ for all } (i, j) \in (n \times m) \text{ imply } x = y,$$

$$(C6) \quad \sigma_{ij}x \vee \sim \sigma_{ij}x = 1,$$

$$(C7) \quad \sigma_{ij}(\sim x) = \sim \sigma_{(n-i)(m-j)}x.$$

The results announced here for $LM_{n \times m}$ -algebras will be used throughout the paper.

(LM1) $\sigma_{ij}(L) = B(L)$ for all $(i, j) \in (n \times m)$, where $B(L)$ is the set of all Boolean elements of L ([15, Proposition 2.5]).

(LM2) Every $LM_{n \times 2}$ -algebra is isomorphic to an n -valued Łukasiewicz-Moisil algebra. It is worth noting that $LM_{n \times m}$ -algebras constitute a non-trivial generalization of the latter (see [18, Remark 2.1]).

(LM3) Let $+$ be the binary operation on L defined as follows:

$$a + b = \bigwedge_{(i,j) \in (n \times m)} ((\sim \sigma_{ij}a \vee \sigma_{ij}b) \wedge (\sim \sigma_{ij}b \vee \sigma_{ij}a)),$$

Then $+$ satisfies the following properties:

$$(T1) \quad a + b = 1 \text{ if and only if } a = b,$$

$$(T2) \quad a + b = b + a,$$

$$(T3) \quad (a + b) \wedge a = (a + b) \wedge b,$$

$$(T4) \quad a + 1 = \sigma_{11}a,$$

$$(T5) \quad \sigma_{rs}(a + b) = a + b, \text{ for every } (r, s) \in (n \times m),$$

$$(T6) \quad \sim(a + b) \text{ and } a + b \text{ are Boolean complements ([15, Proposition 2.6]).}$$

(LM4) The class of $LM_{n \times m}$ -algebras is a variety and two equational bases for it can be found in [15, Theorem 2.7] and [18, Theorem 4.6].

(LM5) Let \rightarrow be the binary operation on L defined as follows:

$$x \rightarrow y = \sigma_{(n-1)(m-1)}(\sim x) \vee y.$$

Then, by defining the notion of deductive system in the usual way, we infer that D is a deductive system of L if and only if D is a Stone filter of L ([15, Proposition 3.3]). Besides, Stone filters were characterized as filters F of L which verify this condition: $x \in F$ implies $\sigma_{11}x \in F$ ([15, Proposition 3.2]). In what follows, we shall denote by $\mathcal{D}(L)$ and $\mathcal{F}_s(L)$ the set of all deductive systems and the set of all Stone filters of L , respectively.

- (LM6) For each $X \subseteq L$, $D(X) = F(\sigma_{11}X)$ ([16, Proposition 2.3.3]), where $D(X)$ and $F(X)$ denote the deductive system and the filter generated by X on L respectively. In particular, if $X = \{a\}$ we shall write $F(a)$ instead of $F(\{a\})$.
- (LM7) Let L be an $LM_{n \times m}$ -algebra with more than one element and let $Con(L)$ be the lattice of all congruences on L . Then $Con(L) = \{R(F) : F \in \mathcal{F}_s(L)\}$, where $R(F) = \{(x, y) \in L \times L : \text{there exists } f \in F \text{ such that } x \wedge f = y \wedge f\}$. Besides, the lattices $Con(L)$ and $\mathcal{F}_s(L)$ are isomorphic considering the mappings $\theta \mapsto [1]_\theta$ and $F \mapsto R(F)$, which are mutually inverse where $[x]_\theta$ stands for the equivalence class of x modulo θ ([15, Theorem 3.6]).
- (LM8) An $n \times m$ -valued Łukasiewicz-Moisil space (or $lm_{n \times m}$ -space) as a trilpe $(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)})$ which verifies the following conditions:
- (E1) (X, g) is a De Morgan space (or m -space) (see [6]),
 - (E2) $f_{ij} : X \rightarrow X$ is a increasing and continuous function,
 - (E3) $f_{ij}(x) \leq f_{(i+1)j}(x)$,
 - (E4) $f_{ij}(x) \leq f_{i(j+1)}(x)$,
 - (E5) $f_{ij} \circ f_{rs} = f_{ij}$,
 - (E6) $f_{ij} \circ g = f_{ij}$,
 - (E7) $g \circ f_{ij} = f_{(n-i)(m-j)}$,
 - (E8) if for every $U, V \in D(X)$ it is verified that $f_{ij}^{-1}(U) = f_{ij}^{-1}(V)$ for all $(i, j) \in (n \times m)$, then $U = V$, where $D(X)$ denotes the lattice of all increasing clopen subsets of X .

Besides, if $(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)})$ and $(X', g', \{f'_{ij}\}_{(i,j) \in (n \times m)})$ are $lm_{n \times m}$ -spaces, an $lm_{n \times m}$ -function is an isotone continuous function $f : X \rightarrow X'$ such that $f \circ g = g' \circ f$ and $f'_{ij} \circ f = f \circ f_{ij}$ for all $(i, j) \in (n \times m)$.

Then, taking into account the topological duality given by W. Cornish and P. Fowler for De Morgan algebras (see [6]), it is proved that the category of $lm_{n \times m}$ -spaces and $lm_{n \times m}$ -functions is naturally equivalent to the dual of the category of $LM_{n \times m}$ -algebras and their corresponding homomorphisms ([19, Theorem 2.1]).

- (LM9) Let X be a non-empty set and let L^X be the set of all functions from X into L . Then L^X is an $LM_{n \times m}$ -algebra where the operations are defined componentwise.
- (LM10) Let $B(L) \uparrow^{(n \times m)} = \{f : (n \times m) \rightarrow B(L) \text{ such that for arbitrarities } i, j \text{ if } r \leq s, \text{ then } f(r, j) \leq f(s, j) \text{ and } f(i, r) \leq f(i, s)\}$. Then

$$\langle B(L) \uparrow^{(n \times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, O, I \rangle$$

is an $LM_{n \times m}$ -algebra where for all $f \in B(L) \uparrow^{(n \times m)}$ and $(i, j) \in (n \times m)$ the operations \sim and σ_{ij} are defined as follows:

$(\sim f)(i, j) = (f(n - i, m - j))'$, where x' denotes the Boolean complement of x ,
 $(\sigma_{ij} f)(r, s) = f(i, j)$ for all $(r, s) \in (n \times m)$, and the remaining operations are defined componentwise ([18, Proposition 3.2]). It is worth noting that this result can be generalized by replacing $B(L)$ by any Boolean algebra B . Furthermore, if B is a complete Boolean algebra, it is simple to check that $B \uparrow^{(n \times m)}$ is also a complete $LM_{n \times m}$ -algebra.

(LM11) Every $LM_{n \times m}$ -algebra L can be embedded into $B(L) \uparrow^{(n \times m)}$ ([18, Theorem 3.1]). Besides, L is isomorphic to $B(L) \uparrow^{(n \times m)}$ if and only if L is centred ([18, Corollary 3.1]) where L is centred if for each $(i, j) \in (n \times m)$ there exists $c_{ij} \in L$ such that

$$\sigma_{rs} c_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s \\ 1 & \text{if } i \leq r \text{ and } j \leq s \end{cases}.$$

2 Monadic $n \times m$ -valued Łukasiewicz-Moisil algebras

The class of algebras which is of our concern now, rises from $n \times m$ -valued Łukasiewicz-Moisil algebras endowed with a unary operation.

Definition 2.1 *A monadic $n \times m$ -valued Łukasiewicz-Moisil algebra (or $mLM_{n \times m}$ -algebra) is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, \exists, 0, 1 \rangle$ where the reduct $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, 0, 1 \rangle$ is an $LM_{n \times m}$ -algebra and \exists is a unary operator on L , called existential quantifier, which verifies these identities:*

$$\begin{aligned} \text{(e1)} \quad \exists 0 &= 0, & \text{(e2)} \quad x \wedge \exists x &= x, \\ \text{(e3)} \quad \exists(x \wedge \exists y) &= \exists x \wedge \exists y, & \text{(e4)} \quad \exists \sigma_{ij} x &= \sigma_{ij} \exists x \text{ for all } (i, j) \in (n \times m). \end{aligned}$$

In what follows we will indicate with $\mathbf{mLM}_{n \times m}$ the class of $mLM_{n \times m}$ -algebras and we shall denote its elements simply by L or (L, \exists) in case we need to specify the quantifier.

Some of the results on $mLM_{n \times m}$ -algebras given in this paper were presented in [7], [14] and [17].

Remark 2.1 *From Definition 2.1 and (LM4) we infer that $\mathbf{mLM}_{n \times m}$ is a variety and two equational bases for it can be obtained.*

Definition 2.2 Let $L \in \mathbf{LM}_{n \times m}$. A universal quantifier on L is a mapping $\forall : L \rightarrow L$ which verifies the following identities:

$$\begin{aligned} (e5) \quad \forall 1 &= 1, & (e6) \quad x \wedge \forall x &= \forall x, \\ (e7) \quad \forall(x \vee \forall y) &= \forall x \vee \forall y, & (e8) \quad \forall \sigma_{ij} x &= \sigma_{ij} \forall x \text{ for all } (i, j) \in (n \times m). \end{aligned}$$

Remark 2.2 (i) Let \exists be an existential quantifier on L . By defining $\forall x = \sim \exists \sim x$ for all $x \in L$, we have that \forall is a universal quantifier on L . Conversely, let \forall be a universal quantifier on L . Then the operator \exists defined by $\exists x = \sim \forall \sim x$ for all $x \in L$ is an existential quantifier on L .

(ii) If $(L, \exists) \in \mathbf{mLM}_{n \times m}$, then from (e4) we have that $(B(L), \exists)$ is a monadic Boolean algebra.

(iii) Taking into account (LM2), we infer that every $mLM_{n \times 2}$ -algebra is isomorphic to a monadic n -valued Lukasiewicz-Moisil algebra.

Lemma 2.1 summarizes the most important properties of both existential and universal quantifiers which are necessary for further development. Its proof is an easy exercise.

Lemma 2.1 Let $L \in \mathbf{mLM}_{n \times m}$. Then the following properties are satisfied:

$$\begin{aligned} (e9) \quad \exists 1 &= 1, & (e10) \quad \exists \exists x &= \exists x, \\ (e11) \quad x \in \exists L &\text{ if and only if } \exists x = x, & (e12) \quad x \leq y &\text{ implies } \exists x \leq \exists y, \\ (e13) \quad \exists(x \vee y) &= \exists x \vee \exists y, & (e14) \quad \exists \sim \sigma_{ij} \exists x &= \sim \sigma_{ij} \exists x, \\ (e15) \quad \forall 0 &= 0, & (e16) \quad \forall \forall x &= \forall x, \\ (e17) \quad x \leq y &\text{ implies } \forall x \leq \forall y, & (e18) \quad x \in B(L) &\text{ implies } \forall x \in B(L), \\ (e19) \quad \forall(x \wedge y) &= \forall x \wedge \forall y, & (e20) \quad \forall \sim \sigma_{ij} \forall x &= \sim \sigma_{ij} \forall x, \\ (e21) \quad \exists \forall x &= \forall x, \quad \forall \exists x = \exists x, & (e22) \quad x = \forall x &\text{ if and only if } x = \exists x, \\ (e23) \quad \exists \sim \sigma_{ij} \forall x &= \sim \sigma_{ij} \forall x, & (e24) \quad \forall \sim \sigma_{ij} \exists x &= \sim \sigma_{ij} \exists x. \end{aligned}$$

Propositions 2.1 and 2.2 determine the relationship between existential quantifiers and special subalgebras of $mLM_{n \times m}$ -algebras.

Proposition 2.1 Let $L \in \mathbf{mLM}_{n \times m}$. Then

- (i) $\exists(L)$ is a Moore family of L and $\exists x = \bigwedge \{z \in \exists(L) : x \leq z\}$, where $\bigwedge X$ denotes the infimum of all elements of the set X ,
- (ii) $\exists(L)$ is a subalgebra of L ,

- (iii) for each $x \in L$, $\sigma_{ij}(\bigwedge\{z \in \exists(L) : x \leq z\}) = \bigwedge\{z \in \exists(L) : \sigma_{ij}x \leq z\}$, for all $(i, j) \in (n \times m)$,
- (iv) if $x, y \in \exists(L)$ and there exists $x \Rightarrow y$ in L , then $x \Rightarrow y \in \exists(L)$, where $a \Rightarrow b$ stands for the relative pseudocomplement of a with respect to b .

Proof. From (e2), (e10) and (e13) we have that \exists is an additive closure operator and taking into account the well-known relationship between closure operators and Moore families (see [1]) we conclude that (i) holds. On the other hand, it is straightforward to prove that (ii) is verified. Besides, (iii) follows from (e4). In order to prove (iv), suppose that $x, y \in \exists(L)$. Then $\exists((x \Rightarrow y) \wedge \exists x) = \exists(x \wedge (x \Rightarrow y)) = \exists(x \wedge y) \leq \exists y = y$. This inequality and (e3) imply that $x \wedge \exists(x \Rightarrow y) \leq y$ and so, $\exists(x \Rightarrow y) \leq x \Rightarrow y$. Therefore, from (e2) we conclude that $\exists(x \Rightarrow y) = x \Rightarrow y$ and the proof is complete. \square

The following is a partial converse of Proposition 2.1.

Proposition 2.2 *Let $L \in \mathbf{LM}_{n \times m}$ and let M be a subset of L which verifies the following conditions:*

- (i) M is a Moore family of L ,
- (ii) M is a subalgebra of L ,
- (iii) for each $x \in L$, $\sigma_{ij}(\bigwedge\{z \in M : x \leq z\}) = \bigwedge\{z \in M : \sigma_{ij}x \leq z\}$, for all $(i, j) \in (n \times m)$,
- (iv) for each $x, y \in M$, there exists $x \Rightarrow y$ in L and $x \Rightarrow y \in M$.

For each $x \in L$ we define $\exists x = \bigwedge\{z \in M : x \leq z\}$. Then \exists is an existential quantifier on L and $M = \exists(L)$.

Proof. From the hypothesis, it is simple to verify that conditions (e1), (e2) and (e4) hold and that $M = \exists(L)$. Besides, from (e2) we have that $x \wedge \exists y \leq \exists x \wedge \exists y$. By virtue of (ii), it results that $\exists x \wedge \exists y \in M$. Hence, $\exists(x \wedge \exists y) \leq \exists x \wedge \exists y$. On the other hand, if $k \in M$ verifies that $x \wedge \exists y \leq k$, then $x \leq \exists y \Rightarrow k$. Furthermore, from (iv) we infer that $\exists y \Rightarrow k \in M$. Therefore, $\exists x \leq \exists y \Rightarrow k$ and so, $\exists x \wedge \exists y \leq k$. Thus, $\exists x \wedge \exists y \leq \exists(x \wedge \exists y)$ and consequently, we conclude that (e3) holds. \square

3 Congruences and subdirectly irreducible $mLM_{n \times m}$ -algebras

Now, we will describe the congruence lattices of $mLM_{n \times m}$ -algebras. From now on, we will denote by $Con_m(L)$ the congruence lattice of L .

Definition 3.1 *A Stone filter F of an $mLM_{n \times m}$ -algebra L is a monadic Stone filter (or ms -filter), if F verifies this condition: $x \in F$ implies $\forall x \in F$.*

Definition 3.2 *A monadic deductive system (or $m.d.s.$) of an $mLM_{n \times m}$ -algebra L is a deductive system D satisfying this additional condition: $x \in D$ implies $\forall x \in D$.*

We shall denote by $\mathcal{F}_{ms}(L)$ and $\mathcal{D}_m(L)$ the set of all ms -filters and $m.d.s.$ of L respectively.

Remark 3.1 *From (LM5) we have that the notions of $m.d.s.$ and ms -filter are equivalent.*

Theorem 3.1 *Let L be an $mLM_{n \times m}$ -algebra with more than one element. Then*

- (i) $Con_m(L) = \{R(F) : F \in \mathcal{F}_{ms}(L)\}$, where $R(F)$ is the relation defined in (LM7),
- (ii) *the lattices $Con_m(L)$ and $\mathcal{F}_{ms}(L)$ are isomorphic considering the mappings $\theta \mapsto [1]_\theta$ and $F \mapsto R(F)$, which are mutually inverse.*

Proof. Taking into account (LM5), (LM7) and Remark 3.1, it remains to prove that if $(x, y) \in R(F)$, then $(\exists x, \exists y) \in R(F)$ which is a direct consequence of (e6), (e21), (e3) and the fact that F is an ms -filter. \square

In what follows, for each $F \in \mathcal{F}_{ms}(L)$ we will denote by L/F the quotient algebra of L by $R(F)$.

Remark 3.2 *If $\theta \in Con_m(L)$ and $(x, y) \in \theta$, then it is simple to check that $(x+z, y+z) \in \theta$ for all $z \in L$.*

Next, our attention is focused on characterizing subdirectly irreducible $mLM_{n \times m}$ -algebras. Lemma 3.1 will be fundamental for this purpose.

Lemma 3.1 *Let $L \in mLM_{n \times m}$ and let $H \subseteq L$. If $D_m(H)$ is the $m.d.s.$ generated by H , then $D_m(H) = F(\sigma_{11}\forall H) = D(\forall H)$.*

Proof. By (LM6), it only remains to prove that $D_m(H) = F(\sigma_{11}\forall H)$. Taking into account that $\sigma_{11}x \leq x$ holds for every $x \in L$ and (e6), we infer that $H \subseteq F(\sigma_{11}\forall H)$. Besides, it is simple to check that $F(\sigma_{11}\forall H)$ is a Stone filter and therefore, from (LM5) it is a deductive system. Moreover, if $x \in F(\sigma_{11}\forall H)$ by virtue of (e17), (e19), (e8) and (e16) we have that $\forall x \in F(\sigma_{11}\forall H)$. On the other hand, if T is a m.d.s. of L such that $H \subseteq T$, then $F(\sigma_{11}\forall H) \subseteq T$. Indeed, let $x \in F(\sigma_{11}\forall H)$. Hence, there exist $\sigma_{11}\forall h_1, \dots, \sigma_{11}\forall h_r \in \sigma_{11}\forall H$ such that $\sigma_{11}\forall h_1 \wedge \dots \wedge \sigma_{11}\forall h_r \leq x$. So, from Remark 3.1 we conclude that $x \in T$. \square

On the other hand, for each $L \in \mathbf{mLM}_{n \times m}$, let us consider the $LM_{n \times m}$ -algebra $\exists(L)$, the monadic Boolean algebra $B(L)$ and the Boolean algebra $B(\exists(L))$. Then by defining the mappings

$$\begin{aligned}\gamma_1 : \mathcal{D}_m(L) &\rightarrow \mathcal{D}(\exists(L)), \quad \gamma_1(D) = D \cap \exists(L), \\ \gamma_2 : \mathcal{D}_m(L) &\rightarrow \mathcal{F}_m(B(L)), \quad \gamma_2(D) = D \cap B(L), \\ \gamma_3 : \mathcal{D}(\exists(L)) &\rightarrow \mathcal{F}(B(\exists(L))), \quad \gamma_3(D') = D' \cap B(\exists(L)), \\ \gamma_4 : \mathcal{F}_m(B(L)) &\rightarrow \mathcal{F}(B(\exists(L))), \quad \gamma_4(F) = F \cap B(\exists(L)),\end{aligned}$$

where $\mathcal{F}_m(B(L))$ and $\mathcal{F}(B(\exists(L)))$ are the set of monadic filters of $B(L)$ and the set of filters of $B(\exists(L))$ respectively, we infer the following

Theorem 3.2 *Let $L \in \mathbf{mLM}_{n \times m}$. Then the mappings γ_1 , γ_2 , γ_3 and γ_4 are order isomorphisms where $\mathcal{D}_m(L)$, $\mathcal{D}(\exists(L))$, $\mathcal{F}_m(B(L))$ and $\mathcal{F}(B(\exists(L)))$ are ordered by set inclusion. Besides, this diagram commutes*

$$\begin{array}{ccc}\mathcal{D}_m(L) & \xrightarrow{\gamma_1} & \mathcal{D}(\exists(L)) \\ \gamma_2 \downarrow & & \downarrow \gamma_3 \\ \mathcal{F}_m(B(L)) & \xrightarrow{\gamma_4} & \mathcal{F}(B(\exists(L)))\end{array}$$

Proof. Following an analogous reasoning to that given in [4] for n -valued Łukasiewicz-Moisil algebras and from well-known results of the theory of monadic Boolean algebras we have that γ_3 and γ_4 are isomorphisms. On the other hand, from Lemma 3.1 and by applying standard techniques we infer that γ_1 and γ_2 are isomorphisms. Finally, it is straightforward to prove that $\gamma_3 \circ \gamma_1 = \gamma_4 \circ \gamma_2$. \square

By virtue of Theorems 3.2 and 3.1 we are ready to characterize subdirectly irreducible $mLM_{n \times m}$ -algebras.

Theorem 3.3 *Let $L \in \mathbf{mLM}_{n \times m}$. Then the following conditions are equivalent:*

- (i) L is simple,
- (ii) $\exists(L)$ is a simple $LM_{n \times m}$ -algebra,
- (iii) $B(\exists(L))$ is a simple Boolean algebra,
- (iv) L is subdirectly irreducible.

As a consequence of Theorem 3.3 and from well-known results of universal algebra we conclude

Corollary 3.1 *$\mathbf{mLM}_{n \times m}$ is semisimple.*

4 The discriminator variety $\mathbf{mLM}_{n \times m}$

In this section, we will apply the results we developed so far to show that $\mathbf{mLM}_{n \times m}$ is a discriminator variety. Furthermore, we will determine the principal congruences. In what follows, for each $a, b \in L$ we will denote by $\theta(a, b)$ the principal congruence generated by (a, b) .

Recall that the *ternary discriminator* function t on a set A is defined by the conditions

$$t(x, y, z) = \begin{cases} z & \text{if } x = y \\ x & \text{otherwise} \end{cases}.$$

A variety \mathcal{V} is a *discriminator variety*, if it has a polynomial p that coincides with the ternary discriminator function on each subdirectly irreducible member of \mathcal{V} ; such a polynomial is called *ternary discriminator polynomial* for \mathcal{V} .

Theorem 4.1 *$\mathbf{mLM}_{n \times m}$ is a discriminator variety.*

Proof. Let $p(x, y, z) = (\forall(x + y) \wedge z) \vee (\sim \forall(x + y) \wedge x)$. By virtue of (T1) and (e5) we have that $p(x, x, z) = z$. If $x \neq y$, then from (T1) and (e6) we infer that $\forall(x + y) \neq 1$. Moreover, from (T6), (e18) and (e21) it results that $\forall(x + y) \in B(\exists(L))$. So, by Theorem 3.3 we conclude that $\forall(x + y) = 0$. Hence, $p(x, y, z) = x$. \square

Corollary 4.1 *The variety of monadic n -valued Lukasiewicz-Moisil algebras is a discriminator variety.*

Proof. It follows from Theorem 4.1 and (iii) in Remark 2.2. \square

Corollary 4.2 (i) $\mathbf{mLM}_{n \times m}$ is arithmetic,

- (ii) for each $L \in \mathbf{mLM}_{n \times m}$ and $a, b, c, d \in L$, it is verified that $(c, d) \in \theta(a, b)$ if and only if $p(a, b, c) = p(a, b, d)$, i.e. $\mathbf{mLM}_{n \times m}$ has equationally definable principal congruences,
- (iii) every principal congruence on $L \in \mathbf{mLM}_{n \times m}$ is a factor congruence,
- (iv) the principal congruences on $L \in \mathbf{mLM}_{n \times m}$ form a sublattice of the lattice $\text{Con}_m(L)$,
- (v) each compact congruence on $L \in \mathbf{mLM}_{n \times m}$ is a principal congruence,
- (vi) the congruences on each $L \in \mathbf{mLM}_{n \times m}$ are regular,
- (vii) the congruences on each $L \in \mathbf{mLM}_{n \times m}$ are normal,
- (viii) the congruences on each $L \in \mathbf{mLM}_{n \times m}$ are filtral,
- (ix) $\mathbf{mLM}_{n \times m}$ has the congruence extension property.

Proof. It is a direct consequence of Theorem 4.1 and the results established in [21]. \square

Lemma 4.1 will allow us to give a new description of the principal congruences on $\mathbf{mLM}_{n \times m}$ -algebras simpler than the one obtained from (ii) in Corollary 4.2.

Lemma 4.1 Let $L \in \mathbf{mLM}_{n \times m}$. Then

- (i) $\theta(a, b) = \theta(\forall(a + b), 1)$,
- (ii) $[1]_{\theta(a, b)} = F(\forall(a + b))$.

Proof. From Remark 3.2 we can assert that $(a + b, b + b) \in \theta(a, b)$. Taking into account (T1) and (e5) we infer that $(\forall(a + b), 1) \in \theta(a, b)$. On the other hand, from (e6) $(a + b, 1) \in \theta(\forall(a + b), 1)$ and so, from (T3) it results that $(a, b) \in \theta(\forall(a + b), 1)$. Consequently, (i) holds. On the other hand, let $x \in [1]_{\theta(a, b)}$. By virtue of (i) and item (ii) in Corollary 4.2, we have that $p(\forall(a + b), 1, x) = p(\forall(a + b), 1, 1)$ and so, $(\forall(\forall(a + b) + 1) \wedge x) \vee (\sim \forall(\forall(a + b) + 1) \wedge \forall(a + b)) = \forall(\forall(a + b) + 1) \vee (\sim \forall(\forall(a + b) + 1) \wedge \forall(a + b))$. Besides, from (T4), (T6), (LM1) and (e16) we get that $\forall(\forall(a + b) + 1) = \forall(a + b)$. Hence, we infer that $\forall(a + b) \leq x$ and therefore, $[1]_{\theta(a, b)} \subseteq F(\forall(a + b))$. The other inclusion is immediate. \square

Theorem 4.2 *Let $L \in \mathbf{mLM}_{n \times m}$. Then*

$$\theta(a, b) = \{(x, y) \in L \times L : x \wedge \forall(a + b) = y \wedge \forall(a + b)\}.$$

Proof. From Theorem 3.1 and (ii) in Lemma 4.1 we have that

$$\theta(a, b) = R([1]_{\theta(a, b)}) = R(F(\forall(a + b))) = \{(x, y) \in L \times L : x \wedge \forall(a + b) = y \wedge \forall(a + b)\}. \quad \square$$

Remark 4.1 *Taking into account (iii) in Remark 2.2, the principal congruences of monadic n -valued Łukasiewicz-Moisil algebras can be characterized as in Theorem 4.2 by means of the $+$ operation defined in (LM3). By identifying the set $\{1, \dots, n-1\} \times \{1\}$ with $\{1, \dots, n-1\}$ and σ_{i1} with σ_i , $1 \leq i \leq n-1$, the $+$ operation is provided by the formula*

$$a + b = \bigwedge_{i=1}^{n-1} ((\sim \sigma_i a \vee \sigma_i b) \wedge (\sim \sigma_i b \vee \sigma_i a)).$$

The following lemmas will allow us to compute the number of congruences of a finite $\mathbf{mLM}_{n \times m}$ -algebra.

Lemma 4.2 *F is a principal ms -filter of an $\mathbf{mLM}_{n \times m}$ -algebra L if and only if $F = F(\forall(\sigma_{11}a))$, for some $a \in L$.*

Proof. It is routine. \square

Lemma 4.3 *Let $L \in \mathbf{mLM}_{n \times m}$ and $Con_m^P(L)$ the lattice of all principal congruences on L . Then $Con_m^P(L)$ is a Boolean lattice, where*

- (i) $id_L = \theta(1, 1)$,
- (ii) $L \times L = \theta(0, 1)$,
- (iii) $\theta(a, b) \wedge \theta(c, d) = \theta((a + b) \vee \forall(c + d), 1)$,
- (iv) $\theta(\sim \forall(a + b), 1)$ is the Boolean complement of $\theta(a, b)$.

Proof. Let $\mathcal{F}_{ms}^P(L)$ be the set of all principal ms -filters of L . Taking into account (ii) in Theorem 3.1, Lemma 4.1 and (T4) we obtain an isomorphism between $\mathcal{F}_{ms}^P(L)$ and $Con_m^P(L)$ by means of the correspondences $\theta(a, b) \mapsto F(\forall(a + b))$ and $F(\forall\sigma_{11}(a)) \mapsto \theta(a, 1)$, which are mutually inverse. On the other hand, it is simple to verify that $\{1\} = F(\forall\sigma_{11}(1))$, $L = F(\forall\sigma_{11}(0))$, $F(\forall\sigma_{11}a) \wedge F(\forall\sigma_{11}b) = F(\forall\sigma_{11}(a \vee \forall b))$ and $F(\forall\sigma_{11} \sim \forall\sigma_{11}a)$ is the Boolean complement of $F(\forall\sigma_{11}a)$. Hence, the above correspondences allow us to conclude the proof. \square

Lemma 4.4 *Let L be a finite $mLM_{n \times m}$ -algebra and let $a \in L$. Then the following conditions are equivalent:*

- (i) $F(\forall \sigma_{11}(a))$ is a maximal ms -filter of L ,
- (ii) $\forall \sigma_{11}(a)$ is an atom of $B(\exists(L))$.

Proof. It is routine. □

Let $|X|$ stand for the number of elements in a finite set X . As a direct consequence of Lemmas 4.3 and 4.4 we conclude

Proposition 4.1 *Let L be a finite $mLM_{n \times m}$ -algebra and let $\Pi(B(\exists(L)))$ be the set of all atoms of $B(\exists(L))$. Then $|Con_m(L)| = 2^{|\Pi(B(\exists(L)))|}$.*

5 A topological duality for $mLM_{n \times m}$ -algebras

Next, we will show a topological duality for $mLM_{n \times m}$ -algebras taking into account both, the one indicated in (LM8) and that given in [5] for Q -distributive lattices.

Definition 5.1 *$(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)}, E)$ is a monadic $n \times m$ -valued Łukasiewicz space (or $mlm_{n \times m}$ -space) if $(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)})$ is an $lm_{n \times m}$ -space and E is an equivalence relation on X which verifies the following conditions:*

- (ml1) $\nabla_E U \in D(X)$ for each $U \in D(X)$, where $\nabla_E U$ stands for the union of all the equivalence classes to contain an element of U ,
- (ml2) the equivalence classes for E are closed in X ,
- (ml3) for each $U \in D(X)$, $f_{ij}^{-1}(\nabla_E U) = \nabla_E f_{ij}^{-1}(U)$ for all $(i, j) \in (n \times m)$.

Let $(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)}, E)$ and $(X', g', \{f'_{ij}\}_{(i,j) \in (n \times m)}, E')$ be $mlm_{n \times m}$ -spaces. An $mlm_{n \times m}$ -function is an $lm_{n \times m}$ -function $f : X \rightarrow X'$ satisfying $\nabla_E(f^{-1}(U)) = f^{-1}(\nabla_{E'}(U))$, for each $U \in D(X)$.

Proposition 5.1 *Let $(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)}, E)$ be a $mlm_{n \times m}$ -space. Then $m\mathcal{LM}_{n \times m}(X) = (D(X), \sim, \{\sigma_{ij}^X\}_{(i,j) \in (n \times m)}, \exists_E)$ is an $mLM_{n \times m}$ -algebra, where for each $U \in D(X)$, $\sim U = X \setminus g^{-1}(U)$, $\sigma_{ij}^X(U) = f_{ij}^{-1}(U)$ and $\exists_E U$ is the union of all equivalence classes to contain an element of U .*

Proof. From [19, Proposition 2.1] and the results obtained in [5], it only remains to prove (e4) which is a direct consequence of (ml3). □

Proposition 5.2 *Let $(L, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, \exists)$ be an $mLM_{n \times m}$ -algebra and let $X(L)$ be the Priestley space associated with L . Then $\mathbf{mlm}_{n \times m}(L) = (X(L), g_L, \{f_{ij}^L\}_{(i,j) \in (n \times m)}, E_\exists)$ is an $mlm_{n \times m}$ -space, where for each $P \in X(L)$, $g_L(P) = L \setminus \{\sim x : x \in P\}$, $f_{ij}^L(P) = \sigma_{ij}^{-1}(P)$ and $E_\exists = \{(P, Q) \in X(L) \times X(L) : P \cap \exists(L) = Q \cap \exists(L)\}$.*

Proof. Conditions (ml1) and (ml2) are direct consequences of [5]. Besides, from [19, Proposition 2.2] we have that $(X(L), g_L, \{f_{ij}^L\}_{(i,j) \in (n \times m)})$ is an $lm_{n \times m}$ -space. On the other hand, let $U \in D(X(L))$. Then there exists $a \in L$ such that $U = \sigma_L(a) = \{P \in X(L) : a \in P\}$. Therefore, in order to prove (ml3), we must show that $\exists_{E_\exists} f_{ij}^{L-1} \sigma_L(a) = f_{ij}^{L-1} \exists_{E_\exists} \sigma_L(a)$ for all $(i, j) \in (n \times m)$. Indeed, from the hypothesis, (e4) and taking into account that $\sigma_L : L \rightarrow D(X(L))$ is an isomorphism in $\mathbf{LM}_{n \times m}$ and in the variety of all Q -distributive lattices (see [5]), we infer that

$$\exists_{E_\exists} f_{ij}^{L-1} \sigma_L(a) = \exists_{E_\exists} \sigma_L \sigma_{ij}(a) = \sigma_L \exists \sigma_{ij}(a) = \sigma_{ij}^{X(L)} \sigma_L \exists(a) = f_{ij}^{L-1} \exists_{E_\exists} \sigma_L(a). \quad \square$$

Let $\mathbf{mlm}_{n \times m}$ be the category of $mlm_{n \times m}$ -spaces and $mlm_{n \times m}$ -functions and $\mathbf{mLM}_{n \times m}$ the category of $mLM_{n \times m}$ -algebras and their corresponding homomorphisms. Then, from Propositions 5.1, 5.2 and following standard techniques we conclude

Theorem 5.1 *The category $\mathbf{mLM}_{n \times m}$ is naturally equivalent to the dual of the category $\mathbf{mlm}_{n \times m}$.*

Here, taking into account the topological duality above described, we will characterize the lattice of all $mLM_{n \times m}$ -congruences. In order to do this, we introduce the following notion.

Definition 5.2 *Let $(X, g, \{f_{ij}\}_{(i,j) \in (n \times m)}, E)$ be an $mlm_{n \times m}$ -space. A subset Y of X is semimodal if $Y \subseteq f_{ij}^{-1}(Y)$ for every $(i, j) \in (n \times m)$.*

On the other hand, recall that a subset Y of a De Morgan space (X, g) is involutive if $g(Y) = Y$.

From now on, for each $(L, \exists) \in \mathbf{mLM}_{n \times m}$, we will denote by $\mathcal{C}_S(\mathbf{mlm}_{n \times m}(L))$ the set of all closed, involutive and semimodal subsets Y of $\mathbf{mlm}_{n \times m}(L)$ such that $\exists_E(Y) = Y$.

Proposition 5.3 *Let $(L, \exists) \in \mathbf{mLM}_{n \times m}$ and $Y \in \mathcal{C}_S(\mathbf{mlm}_{n \times m}(L))$. Then $\Theta(Y) = \{(x, y) \in L \times L : \sigma_L(x) \cap Y = \sigma_L(y) \cap Y\}$ is a congruence on L .*

Proof. It is an immediate consequence of [19, Proposition 2.3] and the results established in [6] and [5]. \square

Proposition 5.4 *Let $L \in \mathbf{mLM}_{n \times m}$, $\delta \in \text{Con}_m(L)$ and $Y = \{P \in X(L) : [1]_\delta \subseteq P\}$. Then*

- (i) $Y \in \mathcal{C}_S(\mathbf{mIm}_{n \times m}(L))$,
- (ii) $\Theta(Y) = \delta$.

Proof. Taking into account [19, Proposition 2.4] we will only prove that $\exists_E(Y) = Y$. Let $P \in Y$, then $[1]_\delta \subseteq P$. Besides, let $Q \in [P]_{E\exists}$ and let $t \in [1]_\delta$. Then, $\forall t \in [1]_\delta$ and therefore, $\forall t \in P \cap \forall(L) = Q \cap \forall(L)$, from which we infer that $t \in Q$. Hence, $[1]_\delta \subseteq Q$ which implies that $Q \in Y$. So, $\exists_E(Y) \subseteq Y$. The other inclusion is immediate. \square

Theorem 5.2 *Let $L \in \mathbf{mLM}_{n \times m}$. Then the lattice $\mathcal{C}_S(\mathbf{mIm}_{n \times m}(L))$ is isomorphic to the dual lattice $\text{Con}_m(L)$ and the isomorphism is the mapping Θ defined by the prescription $\Theta(Y) = \{(x, y) \in L \times L : \sigma_L(x) \cap Y = \sigma_L(y) \cap Y\}$ (see [6]).*

Proof. It follows from Propositions 5.3 and 5.4 and the results established in [6]. \square

Now, we will determine the elements of $\mathcal{C}_S(\mathbf{mIm}_{n \times m}(L))$ which correspond to principal congruences on L . Since $\theta(a, b) = \theta(a \wedge b, a \vee b)$ there is no loss of generality in assuming $a \leq b$.

Lemma 5.1 *Let $L \in \mathbf{mLM}_{n \times m}$ and let $a, b \in L$ be such that $a \leq b$. Then*

$$\sigma_L(\forall(a + b)) = X(L) \setminus \bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\exists_E(\sigma_L(b) \setminus \sigma_L(a))).$$

Proof. Taking into account [19, Lemma 2.2] we have that $\sigma_L(a + b) = X(L) \setminus \bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))$. Then, from (E6) and (e13) it results that

$$\begin{aligned} \sigma_L(\forall(a + b)) &= \sim \exists_E \sim (X(L) \setminus \bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))) \\ &= \sim \exists_E g_L^{-1}(\bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))) \\ &= \sim \exists_E \bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a)) \\ &= \sim \bigcup_{(i,j) \in (n \times m)} \exists_E f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a)) \end{aligned}$$

$$\begin{aligned}
&= X(L) \setminus g_L^{-1} \left(\bigcup_{(i,j) \in (n \times m)} \exists_E f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a)) \right) \\
&= X(L) \setminus \bigcup_{(i,j) \in (n \times m)} g_L^{-1}(\exists_E f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))) \\
&= X(L) \setminus \bigcup_{(i,j) \in (n \times m)} (g_L^{-1} \circ f_{ij}^{-1})(\exists_E(\sigma_L(b) \setminus \sigma_L(a))) \\
&= X(L) \setminus \bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\exists_E(\sigma_L(b) \setminus \sigma_L(a))).
\end{aligned}$$

□

Theorem 5.3 *Let $L \in \mathbf{mLM}_{n \times m}$ and let $a, b \in L$ be such that $a \leq b$. Then the following conditions are equivalent:*

- (i) $Y = \sigma_L(\forall(a + b))$,
- (ii) $Y \in \mathcal{C}_S(\mathbf{mIm}_{n \times m}(L))$ and $\Theta(Y) = \theta(a, b)$.

Proof. Let $P \in \sigma_L(\forall(a + b))$. Then from Lemma 5.1 we have that $f_{ij}(P) \notin \exists_E(\sigma_L(b) \setminus \sigma_L(a))$ for all $(i, j) \in (n \times m)$ and so, taking into account (E5) we infer that $f_{rs}(P) \notin \bigcup_{(i,j) \in (n \times m)} f_{ij}^{-1}(\exists_E(\sigma_L(b) \setminus \sigma_L(a)))$ for all $(r, s) \in (n \times m)$. Hence, $P \in f_{rs}^{-1}(Y)$ for all $(r, s) \in (n \times m)$. Therefore, Y is semimodal. On the other hand, let $P \in g_L(\sigma_L(\forall(a + b)))$. Then there exists $Q \in \sigma_L(\forall(a + b))$ such that $P = g_L(Q)$. From this last assertion and (E6) we get that $f_{ij}(P) = f_{ij}(Q)$ for all $(i, j) \in (n \times m)$. Furthermore, from Lemma 5.1 we infer that $f_{ij}(Q) \notin \exists_E(\sigma_L(b) \setminus \sigma_L(a))$ for all $(i, j) \in (n \times m)$. Hence, we obtain that $P \in \sigma_L(\forall(a + b))$. Consequently, $g_L(\sigma_L(\forall(a + b))) \subseteq \sigma_L(\forall(a + b))$. The other inclusion follows by taking into account that g_L is involutive. So, Y is involutive. Besides, taking into account that σ_L is an isomorphism and (e21) it results that $\exists_E(Y) = Y$. Since Y is a closed subset of $X(L)$ we conclude that $Y \in \mathcal{C}_S(\mathbf{mIm}_{n \times m}(L))$. From Theorem 4.2 and the fact that σ_L is one-to-one we conclude that $\Theta(\sigma_L(\forall(a + b))) = \theta(a, b)$. The converse is immediate. □

6 Functional representation theorems for $\mathbf{mLM}_{n \times m}$ -algebras

In the present section, we will generalize the results obtained in [9] for monadic n -valued Łukasiewicz–Moisil algebras. More precisely, we will describe three functional representation theorems for $\mathbf{mLM}_{n \times m}$ -algebras pointing out the relationship between them. To this end, the following assertions on monadic Boolean algebras will be necessary.

(H1) A constant of a monadic Boolean algebra L is a Boolean endomorphism c on L such that

$$(c1) \quad c \circ \exists = \exists, \quad (c2) \quad \exists \circ c = c.$$

This mapping has the following properties:

$$(c3) \quad c \circ c = c, \quad (c4) \quad c(x) \leq \exists x \text{ for all } x \in L.$$

In particular, a constant c is a witness to an element z of L if $\exists z = c(z)$, and we will denote it by c_z . Furthermore, a monadic Boolean algebra L is rich if for any $x \in L$ there exists a witness to x .

(H2) Every monadic Boolean algebra is a subalgebra of a rich one ([11, Theorem 11]).

(H3) If L is a monadic Boolean algebra, then there exists a set X and a Boolean algebra B such that

- (i) L is isomorphic to a subalgebra S of the functional Boolean algebra B^X ,
- (ii) for each $f \in S$ there exists $x \in X$ such that $\exists f(x) = f(x)$ ([11, Theorem 12]).

Proposition 6.1 *Let $L \in \mathbf{mLM}_{n \times m}$. Then $\langle B(L) \uparrow^{(n \times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, \exists, O, I \rangle$ is an $mLM_{n \times m}$ -algebra where \exists is defined componentwise and the remaining operations are the ones defined in (LM10).*

Proof. The statement follows from (LM10) and the definition of \exists . □

Theorem 6.1 *Every $mLM_{n \times m}$ -algebra L can be embedded into $B(L) \uparrow^{(n \times m)}$.*

Proof. Taking into account [18, Theorem 3.1], the mapping $\tau : L \rightarrow B(L) \uparrow^{(n \times m)}$ defined by the prescription $\tau(x)(i, j) = \sigma_{ij}x$ for each $x \in L$ and $(i, j) \in (n \times m)$ is a one-to-one $LM_{n \times m}$ -homomorphism. Besides, from (e4) it is simple to check that $\tau(\exists x) = \exists \tau(x)$ for all $x \in L$. □

Corollary 6.1 *Every $mLM_{n \times m}$ -algebra can be embedded into a complete one.*

Proof. This assertion follows from a well-known result on Boolean algebras, (LM10) and Theorem 6.1. □

By defining the notion of centred $mLM_{n \times m}$ -algebras in a similar way to the one given for $LM_{n \times m}$ -algebras and as a direct consequence of (LM11), we conclude

Corollary 6.2 *Let $L \in \mathbf{mLM}_{n \times m}$. Then the following conditions are equivalent:*

- (i) L is centred,
- (ii) L is isomorphic to $B(L) \uparrow^{(n \times m)}$.

Proposition 6.2 *Let B be a complete Boolean algebra and let X be a non-empty set. Then $\langle B \uparrow^{(n \times m)^X}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, \exists, O, I \rangle$ is a complete $mLM_{n \times m}$ -algebra where for all $x \in X$, $(\exists f)(x) = \bigvee f(X)$ for each $f \in B \uparrow^{(n \times m)}$ and the remaining operations are defined componentwise, being $\bigvee f(X)$ the supremum of $f(X) = \{f(y) : y \in X\}$.*

Proof. From (LM9) and (LM10) we have that $B \uparrow^{(n \times m)^X}$ is a complete $LM_{n \times m}$ -algebra and that \exists is well defined on $B \uparrow^{(n \times m)^X}$. Besides, it is simple to check that identities (e1), (e2), (e3) and (e4) hold. \square

For the next functional representation theorem we will apply the results given by P. Halmos for monadic Boolean algebras mentioned at the beginning of this section.

Theorem 6.2 *Let L be an $mLM_{n \times m}$ -algebra. Then there exists a non-empty set X and a Boolean algebra B such that L can be embedded into $(\exists(B)) \uparrow^{(n \times m)^X}$ and $B(L)$ is a subalgebra of B .*

Proof. From (ii) in Remark 2.2, we have that $B(L)$ is a monadic Boolean algebra and so, by (H2) we can assert that $B(L)$ is a subalgebra of a rich monadic Boolean algebra B . Let X be a set of constants of B containing at least one witness to x , for each $x \in B$. Let $\Phi : B \rightarrow (\exists(B))^X$ be the mapping defined by $\Phi(z)(c) = c(z)$ for all $c \in X$. Then Φ is a one-to-one monadic Boolean homomorphism (see [11, Theorem 12]). On the other hand, from (LM10) and (LM9) it results that $(\exists(B)) \uparrow^{(n \times m)^X}$ is an $LM_{n \times m}$ -algebra. Let us consider now the mapping $\Psi : L \rightarrow (\exists(B)) \uparrow^{(n \times m)^X}$ defined by $(\Psi(x)(c))(i, j) = \Phi(\sigma_{ij}x)(c)$ for each $x \in L$, $c \in X$ and $(i, j) \in (n \times m)$. Taking into account the definition of Φ , (C1), (C2), (C3) and (C5) it is simple to verify that Ψ is a one-to-one homomorphism of bounded lattices.

On the other hand, since Φ is a Boolean homomorphism, from (C7) we have that

$$\begin{aligned}
((\Psi(\sim x))(c))(i, j) &= (\Phi(\sigma_{ij}(\sim x)))(c) = (\Phi(\sim \sigma_{(n-i)(m-j)}(x)))(c) \\
&= (\Phi((\sigma_{(n-i)(m-j)}(x)))'(c) = (\Phi(\sigma_{(n-i)(m-j)}(x)))'(c) \\
&= ((\Phi(\sigma_{(n-i)(m-j)}(x)))(c))' = ((\Psi(x)(c))(n-i, m-j))' \\
&= (\sim (\Psi(x)(c)))(i, j) = ((\sim \Psi(x))(c))(i, j),
\end{aligned}$$

for all $c \in X$ and $(i, j) \in (n \times m)$ and therefore, $\Psi(\sim x) = \sim \Psi(x)$. Furthermore, taking into account (C4), we get that

$$\begin{aligned}
((\Psi(\sigma_{rs}x))(c))(i, j) &= (\Phi(\sigma_{ij}\sigma_{rs}x))(c) = (\Phi(\sigma_{rs}x))(c) = ((\Psi(x)(c))(r, s)) \\
&= (\sigma_{rs}((\Psi(x)(c)))(i, j) = ((\sigma_{rs}(\Psi(x)))(c))(i, j),
\end{aligned}$$

for all $c \in X$ and $(i, j) \in (n \times m)$. Hence, $\Psi(\sigma_{rs}x) = \sigma_{rs}\Psi(x)$.

If f belongs to the image of Ψ , then $\Psi(l) = f$ for some $l \in L$. Let g_l be the mapping defined by $g_l(i, j) = \exists \sigma_{ij}l$ for all $(i, j) \in (n \times m)$. It is simple to verify that $g_l \in (\exists(B)) \uparrow^{(n \times m)}$. Besides, from (c4) we have that $f(c)(i, j) = \Phi(\sigma_{ij}l)(c) = c(\sigma_{ij}l) \leq \exists \sigma_{ij}l = g_l(i, j)$ for all $(i, j) \in (n \times m)$ and so, $f(c) \leq g_l$ for all $c \in X$. On the other hand, let $h \in (\exists(B)) \uparrow^{(n \times m)}$ be such that $f(c) \leq h$ for all $c \in X$. Therefore, $f(c)(i, j) \leq h(i, j)$ for each $(i, j) \in (n \times m)$. In particular, $f(c_{\sigma_{ij}l})(i, j) \leq h(i, j)$ and so, $g_l \leq h$. From this last assertion and the fact that g_l is an upper bound of $\{f(c) : c \in X\}$, we infer that $g_l = \bigvee \{f(c) : c \in X\}$. Hence, for each $f \in \Psi(L)$ we define $(\exists f)(c) = \bigvee \{f(c) : c \in X\}$ for all $c \in X$. Moreover, from (c1) and (e4) we have that

$$\begin{aligned} ((\exists(\Psi(x)))(c))(i, j) &= \exists \sigma_{ij}x = c(\sigma_{ij}\exists x) = \Phi(\sigma_{ij}\exists x)(c) \\ &= ((\Psi(\exists x))(c))(i, j), \end{aligned}$$

for all $x \in L$, $c \in X$ and $(i, j) \in (n \times m)$. So, Ψ commutes with \exists . \square

Remark 6.1 (i) Let L be an $mLM_{n \times m}$ -algebra and let X be a non-empty set. There is no loss of generality in assuming that the Boolean algebra $\exists(B(L))$ is complete. Then from Proposition 6.2 we have that $(\exists(B(L))) \uparrow^{(n \times m)^X}$ is a complete $mLM_{n \times m}$ -algebra. Hence, from Theorem 6.2, L can be embedded into a complete $mLM_{n \times m}$ -algebra. It is worth noting that the latter is different from that obtained in Corollary 6.1.
(ii) If all elements of an $mLM_{n \times m}$ -algebra L are Boolean ones, that is to say if $L = B(L)$, then Theorem 6.2 coincides with Halmos's functional representation theorem indicated in (H3).

With the purpose of obtaining the third functional representation theorem, we extend the notion of constant indicated in (H1) to $mLM_{n \times m}$ -algebras as follows:

A constant of an $mLM_{n \times m}$ -algebra L is an $mLM_{n \times m}$ -endomorphism c on L such that $c \circ \exists = \exists$ and $\exists \circ c = c$.

From this definition, it results that $c(L) = \exists(L)$ and therefore, $c : L \rightarrow \exists(L)$ is an $LM_{n \times m}$ -epimorphism such that c is the identity on $\exists(L)$. The notions of witness and rich $mLM_{n \times m}$ -algebras are similar to those given for monadic Boolean algebras.

Lemma 6.1 Let L be a rich $mLM_{n \times m}$ -algebra and let X be a set of constants of L containing at least one witness to x for each $x \in L$. Then the following conditions are equivalent:

- (i) $c(x) = 1$, for all $c \in X$,
- (ii) $x = 1$.

Proof. It is routine. □

Theorem 6.3 *Let L be a rich $mLM_{n \times m}$ -algebra. Then there exists a non-empty set X such that L can be embedded into $(\exists(L))^X$.*

Proof. Let X be a set of constants of L containing at least one witness to x for each $x \in L$. Then X is a non-empty set and from (LM9), we have that $(\exists(L))^X$ is an $LM_{n \times m}$ -algebra. Let $\Omega : L \rightarrow (\exists(L))^X$ be the mapping defined by $\Omega(x)(c) = c(x)$ for all $c \in X$. It is straightforward to prove that Ω is an $LM_{n \times m}$ -homomorphism. Furthermore, Ω is one-to-one. Indeed, let $x, y \in L$ be such that $\Omega(x) = \Omega(y)$. Then $c(x) = c(y)$ for each $c \in X$. Thus, $c(x + y) = 1$, for each $c \in X$ and so, from Lemma 6.1 and (T1) we infer that $x = y$. On the other hand, let $f = \Omega(x)$ for some $x \in L$. Hence, $f(c) = c(x) \leq \exists x$, for each $c \in X$. Besides, if $k \in \exists(L)$ verifies that $f(c) \leq k$ for all $c \in X$, then $f(c_x) \leq k$. From this assertion we have that $\exists x \leq k$ and so, $\bigvee \{f(c) : c \in X\} = \exists x$. For each $f \in \Omega(L)$, we define $(\exists f)(c) = \exists x$ for all $c \in X$, being $f = \Omega(x)$. It is simple to check that $\Omega(\exists y) = \exists(\Omega y)$ for all $y \in L$. Therefore, L is isomorphic to the $mLM_{n \times m}$ -algebra $\Omega(L)$. □

Since every $mLM_{n \times m}$ -algebra is a monadic Boolean algebra whenever $\sigma_{ij}x = x$ for all $(i, j) \in (n \times m)$, then bearing in mind well-known results on monadic Boolean algebras we can assert that there exist $mLM_{n \times m}$ -algebras which are not rich. The next theorem will characterize rich algebras in $\mathbf{mLM}_{n \times m}$.

Theorem 6.4 *Let $L \in \mathbf{mLM}_{n \times m}$. Then the following conditions are equivalent:*

- (i) L is rich,
- (ii) for each $x \in L$ there exists a Stone filter F_x such that the natural map $q_x : L \rightarrow L/F_x$ restricted to $\exists(L)$ is an $LM_{n \times m}$ -isomorphism and $q_x(\exists x) = q_x(x)$.

Proof.

(i) \Rightarrow (ii): Let $x \in L$ and let c_x be a witness to x . Then $F_x = c_x^{-1}(1)$ is a Stone filter of L . Hence, the natural map q_x restricted to $\exists(L)$, which we will denote by $q_x|_{\exists(L)}$, is bijective. Indeed, let $a \in L$, then $c_x(a) = y$ for some $y \in \exists(L)$ and so, $c_x(a) = c_x(y)$. Therefore, $[a]_{R(F_x)} = [y]_{R(F_x)}$. Furthermore, if $z \in \exists(L)$ verifies that $[y]_{R(F_x)} = [z]_{R(F_x)}$, then $c_x(y) = c_x(z)$ from which infer that $y = z$. Thus, each equivalence class has only one element belonging to $\exists(L)$. This assertion allows us to infer that $q_x|_{\exists(L)}$ is an $LM_{n \times m}$ -isomorphism. On the other hand, since $c_x(\exists x) = c_x(x)$, we conclude that $q_x(\exists x) = q_x(x)$.
(ii) \Rightarrow (i): Let $x \in L$. Then there is a Stone filter F_x of L such that $q_x|_{\exists(L)}$ is an $LM_{n \times m}$ -isomorphism and $q_x(\exists x) = q_x(x)$. Let $c = q_x|_{\exists(L)}^{-1} \circ q_x$. Then, $c(\exists a) = q_x|_{\exists(L)}^{-1}(q_x(\exists a)) = \exists a$

for all $a \in L$. Besides, c is an $LM_{n \times m}$ -epimorphism from L onto $\exists(L)$, from which we have that $\exists c(a) = c(a)$ for all $a \in L$. Finally, since $c(x) = q_x|_{\exists(L)}^{-1}(q_x(x)) = q_x|_{\exists(L)}^{-1}(q_x(\exists x)) = \exists x$, we conclude that c is a witness to x . \square

Remark 6.2 Let L be an $LM_{n \times m}$ -algebra and let $[0, b] = \{x \in L : x \leq b\}$, being $b \in B(L)$. Then it is easy to verify that $\langle [0, b], \wedge, \vee, -, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, 0, b \rangle$ is an $LM_{n \times m}$ -algebra, where $-x = \sim x \wedge b$ for each $x \in [0, b]$. Besides, the mapping $h_b : L \rightarrow [0, b]$ defined by $h_b(x) = x \wedge b$ is an $LM_{n \times m}$ -epimorphism whose kernel is $F(b)$.

Corollary 6.3 Let L be a finite $mLM_{n \times m}$ -algebra. Then the following conditions are equivalent:

- (i) L is rich,
- (ii) for each $x \in L$ there exists $b_x \in B(L)$ such that h_{b_x} restricted to $\exists(L)$ is an $LM_{n \times m}$ -isomorphism and $h_{b_x}(\exists x) = h_{b_x}(x)$.

Proof.

(i) \Rightarrow (ii): Let $x \in L$. From Theorem 6.4, there exists a Stone filter F_x such that $q_x : \exists(L) \rightarrow L/F_x$ is an $LM_{n \times m}$ -isomorphism and $q_x(\exists x) = q_x(x)$. Since L is a finite algebra, $F_x = F(b_x)$ for some $b_x \in B(L)$. Then from Remark 6.2, the mapping $\gamma : L/F(b_x) \rightarrow [0, b_x]$ defined by $\gamma([y]_{R(F(b_x))}) = y \wedge b_x$, is an $LM_{n \times m}$ -isomorphism and $h_{b_x} = \gamma \circ q_x$, which allow us to conclude (ii).

(ii) \Rightarrow (i): Let $x \in L$. Then from the hypothesis, there exists $b_x \in B(L)$ from which by Remark 6.2, the mapping $\beta : L/F(b_x) \rightarrow [0, b_x]$ such that $\beta \circ q_{b_x} = h_{b_x}$ is an $LM_{n \times m}$ -isomorphism, being q_{b_x} the natural $LM_{n \times m}$ -homomorphism. Hence, $q_{b_x} = \beta^{-1} \circ h_{b_x}$. From this assertion and (ii), it is straightforward to prove that $q_{b_x}|_{\exists(L)}$ is one-to-one and so, it is an $LM_{n \times m}$ -isomorphism. Besides, from (ii) we infer that $q_{b_x}(\exists x) = q_{b_x}(x)$. Therefore, Theorem 6.4 allows us to conclude that L is rich. \square

Finally, we will show the relationship between the functional representations obtained in Theorems 6.1, 6.2 and 6.3 in the particular case of rich $mLM_{n \times m}$ -algebras.

Let X_L be a set of constants of a rich $mLM_{n \times m}$ -algebra L containing at least one witness to x for each $x \in L$ and let

$$X_{B(L)} = \{c^* = c|_{B(L)} : c \in X_L \text{ and } c \text{ is a witness to at least one } b \text{ for each } b \in B(L)\}.$$

From Theorem 6.3 we have that $\Omega(x) = (c(x))_{c \in X_L}$ for each $x \in L$.

On the other hand, let $\tau^* : L^{X_L} \rightarrow (B(L) \uparrow^{(n \times m)})^{X_L}$ and $P : ((\exists B(L)) \uparrow^{(n \times m)})^{X_L} \rightarrow ((\exists B(L))) \uparrow^{(n \times m)}^{X_{B(L)}}$ be the mappings defined by

$\tau^*((a_c)_{c \in X_L}) = (\tau(a_c))_{c \in X_L}$, being τ the mapping introduced in Theorem 6.1,

$$P((f(c))_{c \in X_L}) = (f(c^*))_{c^* \in X_{B(L)}}.$$

Then for each $x \in L$, we have that

$$\begin{aligned} (P \circ \tau^* \circ \Omega)(x) &= (P \circ \tau^*)((c(x))_{c \in X_L}) = P((\tau(c(x)))_{c \in X_L}) \\ &= P(((\sigma_{ij}(c(x)))_{(i,j) \in (n \times m)})_{c \in X_L}) = P(((c(\sigma_{ij}(x)))_{(i,j) \in (n \times m)})_{c \in X_L}) \\ &= ((c^*(\sigma_{ij}(x)))_{(i,j) \in (n \times m)})_{c^* \in X_{B(L)}} = \Psi(x), \end{aligned}$$

where Ψ is the mapping given in Theorem 6.2. Then the following diagram commutes:

References

- [1] R. Balbes and Ph. Dwinger, *Distributive Lattices*, Univ. of Missouri Press, Columbia, 1974.
- [2] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Lukasiewicz - Moisil Algebras*, Annals of Discrete Mathematics 49, North - Holland, 1991.
- [3] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, Vol. 78, Springer, Berlin, 1981.
- [4] R. Cignoli, *Moisil Algebras*, Notas de Lógica Matemática 27, Univ. Nacional del Sur, Bahía Blanca, Argentina, 1970.
- [5] R. Cignoli, *Quantifiers on distributive lattices*, Discrete Math. 96(1991), 183–197.
- [6] W. Cornish and P. Fowler, *Coproducts of De Morgan algebras*, B. Aust. Math. Soc., Vol. 16, (1977), 1–13.
- [7] A. V. Figallo and C. Sanza, *Advances in monadic $n \times m$ -valued Lukasiewicz algebras with negation*, Abstracts of Lectures, Tutorials and Talks. International Conference on Order, Algebra and Logics. Vanderbilt University, Nashville, USA, (2007), 46.
- [8] A. V. Figallo and C. Sanza, *The $\mathcal{NS}_{n \times m}$ -propositional calculus*, Bulletin of the Section of Logic, 35, 2(2008), 67–79.
- [9] A. V. Figallo, C. Sanza and A. Ziliani, *Functional monadic n -valued Lukasiewicz algebras*, Math. Bohem. 4(2005), 337–348.
- [10] G. Georgescu and C. Vraciu, *Algebre Boole monadice si algebre Lukasiewicz monadice*. Studii Cerc. Mat. 23, 7(1971), 1025–1048.

- [11] Halmos, P., *Algebraic Logic*, Chelsea, New York, 1962.
- [12] Halmos, P., *Lectures on Boolean Algebras*, Van Nostrand, Princeton, 1963.
- [13] Gr. C. Moisil, *Essais sur les logiques non Chrysippiennes*, Bucarest, 1972.
- [14] C. Sanza, *Algebras de Lukasiewicz matriciales $n \times m$ -valuadas con negación monádicas*, Noticiero de la Unión Matemática Argentina 2002, 165.
- [15] C. Sanza, *Notes on $n \times m$ -valued Lukasiewicz algebras with negation*, L. J. of the IGPL, 6, 12(2004), 499–507.
- [16] C. Sanza, *Algebras de Lukasiewicz $n \times m$ -valuadas con negación*. Doctoral Thesis. Universidad Nacional del Sur, Bahía Blanca, Argentina, 2005.
- [17] C. Sanza, *On monadic $n \times m$ -valued Lukasiewicz algebras with negation*. Algebraic and Topological Methods in Non-Classical Logics II. Abstracts, Barcelona, España 2005, 71.
- [18] C. Sanza, *$n \times m$ -valued Lukasiewicz algebras with negation*, Rep. Math. Logic 40(2006), 83–106.
- [19] C. Sanza, *On $n \times m$ -valued Lukasiewicz-Moisil algebras*, Cent. Eur. J. Math., 6(3)(2008), 372–383.
- [20] W. Suchoń, *Matrix Lukasiewicz Algebras*, Rep. on Math. Logic, Vol. 4, (1975), 91-104.
- [21] H. Werner, *Discriminator-Algebras, Algebraic representation and model theoretic properties*, Akademie-Verlag, Berlin, 1978.

A. V. Figallo.

Departamento de Matemática, Universidad Nacional del Sur. 8000 Bahía Blanca. Argentina.

Instituto de Ciencias Básicas, Universidad Nacional de San Juan. 5400 San Juan. Argentina

e-mail: avfigallo@gmail.com

Claudia Sanza.

Departamento de Matemática. Universidad Nacional del Sur. 8000 Bahía Blanca. Argentina.

e-mail: claudiaasanza@gmail.com